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TURBULENT PRESSURE IN THE ENVELOPES OF YELLOW HYPERGIANTS AND LUMINOUS BLUE VARIABLES

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ABSTRACT

The manner in which turbulence (especially turbulent pressure) affects the structure and stability of luminous post–red-supergiant stars is critically evaluated by calculating both realistic and one-zone models of the convective envelopes. In these stars, the remnant outer envelope closely approaches the Eddington limit, with the result that the local gas densities are driven down. Such a tenuous environment promotes high turbulent velocities in the marginally convective layers of the outer envelope. In the hydrogen and helium convection zones, however, the velocities, even though high, fall well below sound velocity, and the temperature gradient there is essentially radiative, making both the turbulent pressure and the turbulent kinetic energy flux structurally unimportant. Instability is tested for by assuming that turbulence adapts either slowly or rapidly to small perturbations, depending on the magnitude of the turbulent velocity. Although the adiabatically stratified iron convection zone lies too deep below the surface to influence the formal dynamical stability or instability of the outer envelope, radiative instability in this zone is increased if supersonic turbulence occurs and generates energetic shocks or if convection is unable to transport all of the super-Eddington luminous flux. It is concluded that turbulent pressure has no significant effect on the formal dynamical instability of the outer envelope in yellow hypergiant stars and luminous blue variables (LBVs), but it may significantly ease the requirement for radiative instability in the brightest and hottest LBVs and in their close relatives, the hydrogen-poor WN stars. Since both dynamical instability and radiative instability lead to a strong dynamical outflow of matter, the unresolved complications arising from supersonic turbulence and from the consequent inapplicability of mixing-length theory render uncertain the predicted domains of instability for the brightest and hottest stars.

Subject headings: stars: mass loss — stars: oscillations — stars: variables: other — stars: Wolf-Rayet — turbulence

1. INTRODUCTION

A hot, massive, post–main-sequence star that has lost most of its hydrogen-rich envelope presents a bloated appearance owing to its enormous internal radiation pressure. Inside a star like this resides a massive compact core surrounded by little more than a huge bubble of blackbody radiation. Since radiation pressure tends to decrease stellar stability, the outer envelope (containing layers cooler than $\sim 2 \times 10^5$ K) hovers perpetually near the borderline of convective, radiative, and dynamical instability (Stothers & Chin 1983, 1993). Small contributions of atomic continuum and line opacity, rising above the uniform electron-scattering background that dominates in a low-density, high-temperature environment, are able to tip the balance in favor of any of these three major types of instability. In addition, low-amplitude pulsational instabilities may also be triggered by the tiny bumps of atomic opacity (Kiriakidis, Fricke, & Glatzel 1993; Glatzel et al. 1999; Dorfi & Gautschy 2000).

Whether low-amplitude pulsations can grow sufficiently to eject matter (Langer et al. 1994; Guzik et al. 1997) is an uncertain conjecture, because driving of any outbursts probably depends on nearly quenching convection in the iron opacity bump region of the star during certain phases of the pulsation cycle. With convection nearly suppressed, the local super-Eddington luminosity is then able to rapidly expel the overlying layers. This outcome can happen even without any pulsations if the stellar wind mass-loss rate is high enough, because in that case the mass-loss acceleration throughout the outer envelope reduces the effective gravity so much that the luminosity can exceed the Eddington limit in most of the outer envelope layers (Kato & Iben 1992; Stothers 2002a).

The importance of the iron convection zone in hot massive stars should not obscure the fact that additional convection zones, arising from the partial ionizations of hydrogen and helium, exist closer to the surface. Convection in these very diffuse zones, however, is extremely inefficient at transporting energy, which is therefore transferred almost entirely by radiation. Sometimes these outer zones can be adequately approximated as being purely radiative. Such an assumption, however, is valid only if one ignores the dynamical effects of convection. Application of mixing-length theory shows that deep in the convection zones of the hottest stars the mean velocity of turbulent elements approaches the velocity of sound and can even become supersonic if the effective gravity is low enough (Stothers 2002a). This produces a large local turbulent pressure, which should not be ignored.

In the present paper, the effects of including turbulent pressure in models of blue and yellow, massive, post–main-sequence stars are investigated for the first time. The equilibrium structure of the outer envelope, its radiative stability, and its dynamical stability are considered. Observationally, the models are expected to describe stars such as yellow hypergiants, luminous blue variables (LBVs or S Doradus variables), and hydrogen-poor Wolf-Rayet stars (WN stars). These objects are taken here to be post–red-supergiant stars, as discussed elsewhere (de Jager et al. 2001;

Stothers & Chin 2001; Stothers 2002a). The alternative interpretation, that they (or at least the most luminous of them) have never been red supergiants, has been presented by Langer et al. (1994), Pasquali et al. (1997), and Lamers et al. (2001).

In §§ 2 and 3 the equation of hydrostatic equilibrium and the dynamical departures from equilibrium are treated in a mathematically general way. Applications of the theory to yellow hypergiants in § 4 and to extreme blue supergiants in § 5 follow. Radiative instability arising from a super-Eddington luminosity in the hottest supergiants is also discussed. A summary of our main results concludes the paper in § 6.

2. HYDROSTATIC EQUILIBRIUM

In the simplest models of stellar convection zones, the conventional equation of hydrostatic equilibrium (gradient of the gas and radiation pressure balances the force of gravity) is employed, and the total heat flux is taken to be the sum of the radiative and convective heat fluxes. All of the standard convective quantities, including the true gradient of temperature, are evaluated from simple mixing-length theory. Other turbulence phenomena—which are traditionally ignored—introduce three additional terms into the basic equations: a turbulent pressure, a turbulent kinetic energy flux, and a buoyancy force (Canuto 1993).

The turbulent term that is most often referred to in the literature is the turbulent pressure. This is given by

$$P_{\rm turb} = C_{\rm turb}\rho v_{\rm turb}^2 \ , \qquad (1)$$

where ρ is the mass density, $v_{\rm turb}$ is the mean turbulent velocity, and $C_{\rm turb} = 1/3$ for isotropic turbulence. Since the buoyancy force is extremely small compared to gravity under most conditions (V. M. Canuto 2002, private communication), it can be omitted. This leaves us with the turbulent kinetic energy flux, which has to be added to the radiative and convective fluxes in order to make up the total flux. Although often substantial in size, the turbulent kinetic energy flux, like the convective flux, plays no significant role in determining the structure of a hot, luminous stellar envelope. In the hydrogen and helium convection zones near the surface, the density is so low that nearly all of the luminosity is transported by radiation and therefore the temperature gradient remains close to the radiative one. At greater depths, in the iron convection zone, the temperature gradient is essentially equal to the adiabatic one. The atomic partial ionizations that cause convection to break out occur at roughly the following temperatures: H^+, 8×10^3 K; He^+, 1.2×10^4 K; He^{++}, 3×10^4 K; and Fe ionization, 1.5×10^5 K.

Thus, the structure of the stellar envelope will be significantly affected by turbulence only through the turbulent pressure in the equation of hydrostatic equilibrium. This equation can be written

$$\frac{1}{\rho}\frac{d}{dr}(P_{\rm gas} + P_{\rm rad} + P_{\rm turb}) = -g + f \ , \qquad (2)$$

where the density ρ is given by

$$\rho = \frac{1}{4\pi r^2}\frac{dM(r)}{dr} \ , \qquad (3)$$

$g = GM(r)/r^2$ is the gravitational acceleration, and f is the stationary mass-loss acceleration of the outwardly moving envelope (the stellar wind). In this paper we take $f/g =$ constant for simplicity, an approximation that has been justified elsewhere (Stothers 2002a).

Generally speaking, the envelope structure becomes altered by the turbulent pressure in just one important way. The additional pressure means that less gas and radiation pressure is needed to balance gravity. The consequence is a *lower density* (see also Stellingwerf 1976; Jiang & Huang 1997). Another consideration is that, if $P_{\rm turb}$ is treated thermodynamically as a component of the total pressure P rather than hydrodynamically as a part of the total work done against the effective gravity, the adiabatic temperature gradient also becomes modified by the turbulent pressure. Such a modification will affect the envelope structure, but only in the deeper, adiabatic layers, for an assigned set of surface parameters (M, L, T_e, etc.). In the next section, we consider these two possible treatments of $P_{\rm turb}$.

3. DYNAMICAL INSTABILITY

If the envelope is rapidly perturbed, the dynamical acceleration, d^2r/dt^2 (which is not to be confused with the stationary mass-loss acceleration due to the stellar wind, f), must be added to equation (2). The resulting equation of motion is

$$\frac{d^2r}{dt^2} = -\frac{1}{\rho}\frac{d}{dr}(P_{\rm gas} + P_{\rm rad} + P_{\rm turb}) - g + f \ . \qquad (4)$$

How should the turbulent pressure variations arising from the imposed perturbation be treated? They can be formally included either hydrodynamically (§ 3.1) or thermodynamically (§ 3.2).

3.1. *Slowly Adapting Turbulence*

If turbulence reacts very slowly to the perturbation, it can be assumed not to adjust at all. This will be the case if the time taken by a turbulent element to move through a mixing length l is very long compared to the equivalent time for a sound wave to travel, in other words, if the ratio $v_{\rm turb}/v_{\rm sound}$ is very small. As an approximation, equation (4) can then be rewritten

$$\frac{d^2r}{dt^2} = -\frac{1}{\rho}\frac{dP}{dr} - g_{\rm eff} \ , \qquad (5)$$

with

$$P = P_{\rm gas} + P_{\rm rad} \ , \qquad (6)$$

$$g_{\rm eff} = g - f + 4\pi r^2\frac{dP_{\rm turb}}{dM(r)} \ . \qquad (7)$$

Here $P_{\rm turb}$, and therefore $dP_{\rm turb}/dM(r)$, is assumed to be constant in time. In an actual stellar convection zone, $P_{\rm turb}$ rises to a maximum from zero at both boundaries, and therefore the sign of $dP_{\rm turb}/dM(r)$ changes from negative to positive in going downward through the zone.

Consider now a small radial perturbation of all the variables, of the form $r = r_0 + \delta r\exp(i\sigma t)$, where a zero subscript refers to the equilibrium state and σ is a complex pulsation frequency. If the perturbations are purely adiabatic, as is appropriate for the consideration of dynamical instability (Stothers 1999), then $\delta P/P_0 = \Gamma_1 \delta\rho/\rho_0$. Introducing the perturbed variables into equations (3) and (5) and linearizing, we find

$$\frac{d^2}{dr^2}\left(\frac{\delta r}{r}\right) + \left(\frac{4 - V + C}{r}\right)\frac{d}{dr}\left(\frac{\delta r}{r}\right) + \frac{V}{\Gamma_1 r^2}\left[\frac{\sigma^2 r^3}{GM(r)}\left(\frac{g}{g_{\rm eff}}\right) - (3\Gamma_1 - 4) + 4\left(\frac{P_{\rm turb}}{P}\right)\frac{d\ln P_{\rm turb}}{d\ln P} + \frac{3\Gamma_1 C}{V}\right]\frac{\delta r}{r} = 0 \ , \quad (8)$$

where $V = -(d\ln P)/(d\ln r)$, $C = (d\ln\Gamma_1)/(d\ln r)$, and the zero subscripts have been dropped.

The necessary and sufficient condition for dynamical instability is that $\sigma^2 \le 0$, where σ is the smallest eigenvalue for which $\delta r/r$ is finite at the surface and zero at the base of the envelope. To acquire a feel for how turbulence affects dynamical instability under the constraint of a very slow turbulent adaptation, consider the one-zone model of a stellar envelope with a constant value of Γ_1. Then equation (8) immediately yields the equivalent criterion,

$$\Gamma_1 \le \frac{4}{3}\left(1 + \frac{P_{\rm turb}}{P}\frac{d\ln P_{\rm turb}}{d\ln P}\right) . \quad (9)$$

Since the sign of $dP_{\rm turb}/dP$ depends on what part of the convection zone is being represented by the one-zone model, the effect of turbulent pressure can be *either stabilizing or destabilizing*. In the absence of turbulence, Γ_1 must drop below a very small value, 4/3; this will normally be achieved by the partial ionizations of hydrogen and helium, because a neutral or a fully ionized gas has $\Gamma_1 = 5/3$ and blackbody radiation alone can bring Γ_1 down only as far as 4/3. With turbulence included, the maximum value of Γ_1 to still have instability can thus be either smaller or greater than 4/3.

The foregoing considerations suggest that it may be convenient to express the turbulent pressure as a ratio, $P_{\rm turb}/P$. To gain a more transparent representation, we use the ratio of the mean turbulent velocity, $v_{\rm turb}$, to the adiabatic velocity of sound,

$$v_{\rm sound} = \left(\frac{\Gamma_1 P}{\rho}\right)^{1/2} , \quad (10)$$

in other words the Mach number, so that

$$\frac{P_{\rm turb}}{P} = C_{\rm turb}\Gamma_1\left(\frac{v_{\rm turb}}{v_{\rm sound}}\right)^2 . \quad (11)$$

3.2. *Rapidly Adapting Turbulence*

A rapid response of turbulence to an imposed perturbation might take any number of forms. As an extreme case, we assume that the pressure and the specific kinetic energy of the turbulent eddies behave thermodynamically like those of a gas (not necessarily classical). Subject to certain mathematical constraints, the thermodynamic relations for turbulence can then be included with those for an ionizing gas and for blackbody radiation to derive generalized specific heats and generalized adiabatic exponents.

Under these circumstances, the equation of motion (eq. [5]) applies if the following definitions are used:

$$P = P_{\rm gas} + P_{\rm rad} + P_{\rm turb} \ , \quad (12)$$

$$g_{\rm eff} = g - f \ . \quad (13)$$

Linearizing as before, we get

$$\frac{d^2}{dr^2}\left(\frac{\delta r}{r}\right) + \left(\frac{4 - V + C}{r}\right)\frac{d}{dr}\left(\frac{\delta r}{r}\right) + \frac{V}{\Gamma_1 r^2}\left[\frac{\sigma^2 r^3}{GM(r)}\left(\frac{g}{g_{\rm eff}}\right) - (3\Gamma_1 - 4) + \frac{3\Gamma_1 C}{V}\right]\frac{\delta r}{r} = 0 \ . \quad (14)$$

All of the turbulent effects have now been subsumed under the generalized expression for Γ_1, which contains the ratio $\delta = P_{\rm turb}/(P_{\rm gas} + P_{\rm rad})$ in addition to the usual ratio $\beta = P_{\rm gas}/(P_{\rm gas} + P_{\rm rad})$ (Stothers 2002b). Dynamical instability in the one-zone stellar envelope model occurs if $\Gamma_1 \le 4/3$, which is unchanged from the usual criterion. For the present case of rapidly adapting turbulence, we may regard the turbulent eddies as behaving like a "classical gas" and assign $C_{\rm turb} = 1/3$ and $\sigma_T = (\partial\ln v_{\rm turb}/\partial\ln T)_\rho = 1/2$ in the generalized formula for Γ_1. Since turbulence for this case tends to increase Γ_1 (because $\Gamma_1 \to 5/3$ as $\delta \to \infty$), its effect on the envelope will be a *stabilizing* one.

Although the ratio $P_{\rm turb}/P$ does not occur explicitly in the above expressions, it can be represented in terms of the Mach number by equations (10) and (11) with $P = P_{\rm gas} + P_{\rm rad} + P_{\rm turb}$.

It is clear from the general discussion of dynamical instability in this section that slowly adapting turbulence can be either stabilizing or destabilizing, while rapidly adapting turbulence is always stabilizing. In a star such as the Sun, where the lifetime of a convective eddy is several weeks but the pulsational timescale is only minutes, turbulence must adapt very slowly. On the other hand, inside the iron convection zone of a very blue LBV, turbulence is supersonic and therefore must adapt very quickly. For an intermediate situation in which the turbulent velocity approaches close to, yet does not exceed, the velocity of sound, the choice is not clear-cut. Other relevant factors include the total thickness of the convection zone and the depth of the bottom of the region affected by dynamical instability. A one-zone stellar model is insufficient to treat these spatial factors. In the next two sections, applications to realistic models of yellow and blue luminous supergiants are made.

4. YELLOW HYPERGIANTS

Yellow hypergiants are thought to be stars that were formerly luminous red supergiants and are now attempting to cross the Hertzsprung-Russell (H-R) diagram (de Jager 1998). Affected by a number of atmospheric and envelope instabilities, they probably suffer most fundamentally from classical dynamical instability in the outer envelope (de Jager et al. 2001).

The blue edge of the theoretical region of dynamical instability for yellow hypergiants has been calculated previously without the inclusion of either turbulent pressure or mass-loss acceleration in the stellar envelope (Stothers & Chin 2001). Nevertheless, the predicted blue edge matches very well the observed edge of the domain of yellow hypergiants. Why is this so?

The mass-loss acceleration is given by (Stothers 2002a)

$$f = \left(\frac{h}{\delta M}\frac{dM}{dt}\right)^2 \frac{R^3 g}{GM} , \qquad (15)$$

where h is taken to be a constant equal to 10. For stellar models located along the blue edge, f turns out to be roughly 5 orders of magnitude smaller than g, primarily because the mass of the outer envelope, δM, is very large for such cool stars. These stars have $\delta M/M = 10^{-4}$ to 10^{-5}, compared to 10^{-7} to 10^{-8} for LBVs. Therefore, f can be safely ignored in calculating models for yellow hypergiants, and we set $f/g = 0$.

As for the turbulent pressure, a more careful analysis is required. Four models of stellar envelopes lying along the blue edge have been recomputed in order to evaluate the effects of including $P_{\rm turb}$ in the case of slowly adapting turbulence. The original stellar models come from our most recent study (Stothers 2002a) and cover the luminosity range $\log(L/L_\odot) = 5.5$–6.1. Masses of the stars are 10.4, 15.4, 21.6, and 34.6 $M_\odot$, or roughly one-third of the initial masses of the stars on the main sequence.

Attention is first focused on the envelope model for a yellow hypergiant of 21.6 $M_\odot$ with $\log T_e = 3.97$, as a typical example. Figure 1 displays the interior progression with temperature of four physical variables running from the photosphere down into the iron opacity bump region. Notice the rather flat distribution of the density that is interrupted by two small inversions in the helium and iron convection zones, caused by mild local maxima of the opacity κ. The ratio of the mean turbulent velocity to the adiabatic sound velocity, $v_{\rm turb}/v_{\rm sound}$, is also shown. The turbulent velocity has been computed by assuming $v_{\rm turb} = v_{\rm conv}$, where $v_{\rm conv}$ is the mean convective velocity derived from standard mixing-length theory (Böhm-Vitense 1958; Cox & Giuli 1968). We adopt a ratio of mixing length to local pressure scale height, α_P, equal to 1.4. Although this value of α_P is based on a calibration of the effective temperatures of very luminous red supergiants (Stothers & Chin 1997), we take it to apply also to very luminous yellow and blue supergiants.

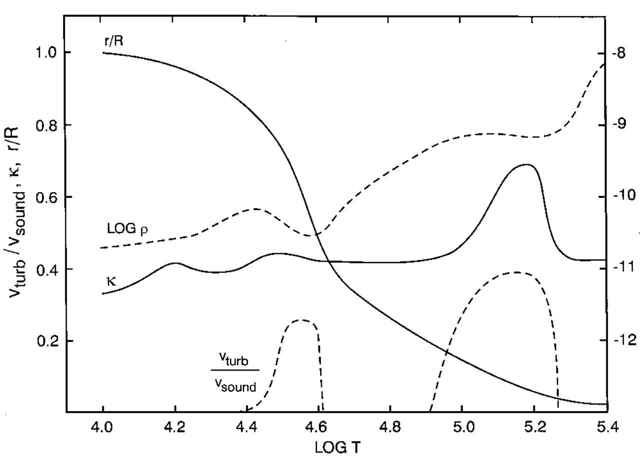

FIG. 1.—Runs of several physical variables through the outer envelope of a yellow hypergiant model with $M/M_\odot = 21.6$, $\log(L/L_\odot) = 5.802$, $\log T_e = 3.97$, $\alpha_P = 1.4$, and $f/g = 0$. Units of the variables are either dimensionless or cgs. The model sits on the borderline of dynamical instability.

If $\alpha_P = 1.4$, then $v_{\rm turb}/v_{\rm sound}$ remains below 0.01, 0.3, and 0.4 in the hydrogen, helium, and iron convection zones, respectively. With $C_{\rm turb} = 1/3$, it is clear that $P_{\rm turb}$ stays everywhere less than 7% of the combined pressure of gas and radiation and also affects only a limited volume of the outer envelope.

A smaller value of α_P would imply less efficient convection, rendering $P_{\rm turb}$ even more insignificant. At first sight, it might be thought that α_P could, alternatively, be larger. Low-luminosity red supergiants have their effective temperatures matched best by models with $\alpha_P = 2.8$ (Stothers & Chin 1997), and local fits of α_P to the results of numerical simulations of convection in the Sun have ranged from ~ 0 near the surface up to ~ 3 in the adiabatic interior (Kim et al. 1996; Demarque, Guenther, & Kim 1997; Porter & Woodward 2000). However, the important helium and iron convection zones in the present yellow and blue supergiant models extend over only 2–3 pressure scale heights, and therefore we regard 1.4 as probably an upper limit for α_P in calculating $v_{\rm turb}$.

Instead of subsuming all the uncertainty about the magnitude of $P_{\rm turb}$ under the parameter α_P, we could arbitrarily change the value of $C_{\rm turb}$ (Rosenthal et al. 1999). Since turbulence must possess some anisotropy, $C_{\rm turb}$ cannot be exactly 1/3. Part of the anisotropy arises from the expected upflows and downflows in turbulent stellar convection zones (Li et al. 2002), which would make $C_{\rm turb} > 1/3$, but a greater part probably comes from the restricted vertical size of the present supergiant convection zones, which would make $C_{\rm turb} < 1/3$. If the star is pulsating, convection might be partly suppressed, further reducing the size of the convection zones and decreasing the effective value of $C_{\rm turb}$.

Another legitimate question is whether $v_{\rm turb} = v_{\rm conv}$, as assumed. This assumption is probably a reasonable one, judging from general arguments about turbulent flows inside stars (Cox & Giuli 1968, pp. 288, 302) as well as from detailed analytic models of stellar turbulence including the full spectrum of eddy sizes (Canuto & Mazzitelli 1991) and, especially, from detailed numerical simulations of solar envelope convection with output velocity data fitted to the mixing-length equations (Abbett et al. 1997; Ludwig, Freytag, & Steffen 1999; Li et al. 2002). The numerical results actually show that $v_{\rm turb}$ is slightly smaller than $v_{\rm conv}$. All of these considerations suggest that by adopting $\alpha_P = 1.4$ and $P_{\rm turb} = (1/3)\rho v_{\rm conv}^2$ we have most likely *overestimated* the turbulent pressure in our present stellar models.

Finally, it should not be expected that a local theory like mixing-length theory can yield accurate values of the *gradient* of $P_{\rm turb}$, especially near the boundaries of the convection zone where convective overshooting must occur. In these layers, however, $P_{\rm turb}$ is small, and so $dP_{\rm turb}/dP$ must be too.

The situation regarding turbulence is essentially identical for our three other models of yellow hypergiants lying along the blue edge. Since the blue edge rises more or less vertically on the H-R diagram, changes of stellar mass and luminosity do not lead to any alteration of our basic conclusions. Even if the absolute locus of the blue edge's effective temperature remains a bit uncertain (Stothers & Chin 2001), Figure 2 shows that the peak values of $v_{\rm turb}/v_{\rm sound}$ stay low over a wide range of effective temperatures, 5000–20,000 K. The effect of $P_{\rm turb}$ on the blue edge is small, increasing $\log T_e$ by less than 0.1. The net destabilizing effect has some uncertainty, however, owing to the large changes of gradient,

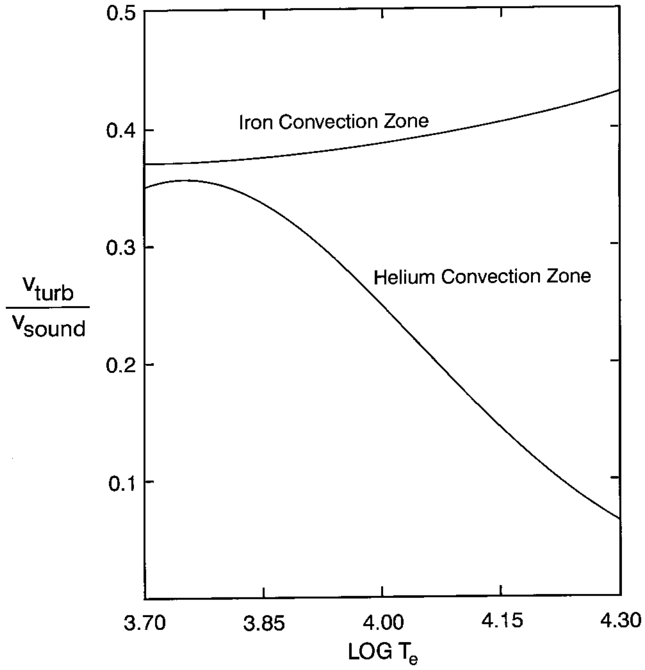

FIG. 2.—Peak Mach number (ratio of mean turbulent velocity to adiabatic sound velocity) in the helium convection zone and in the iron convection zone, as a function of effective temperature. The ratios refer to post–red-supergiant models with $M/M_{\odot} = 21.6$, $\log(L/L_{\odot}) = 5.802$, $\alpha_P = 1.4$, and $f/g = 0$.

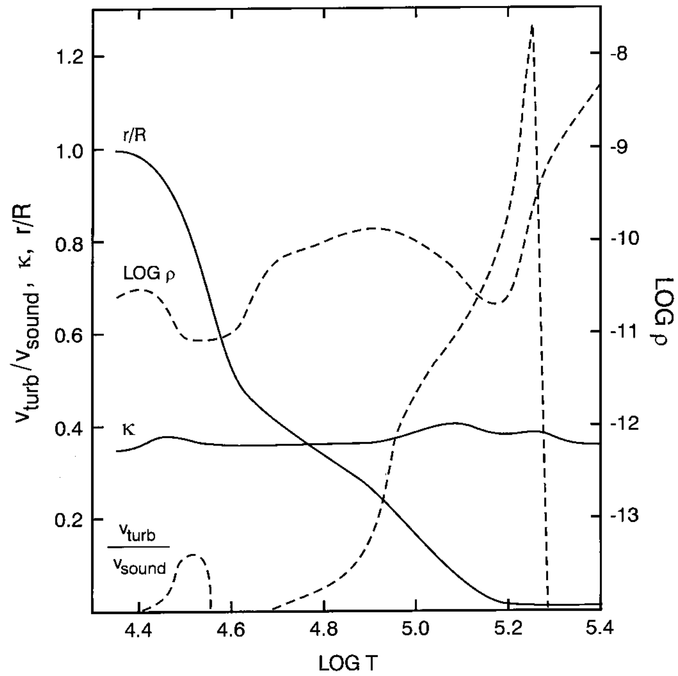

FIG. 3.—Runs of several physical variables through the outer envelope of an LBV model with $M/M_{\odot} = 21.6$, $\log(L/L_{\odot}) = 5.802$, $\log T_e = 4.35$, $\alpha_P = 1.4$, and $f/g = 0.16$. Units of the variables are either dimensionless or cgs. The model sits on the borderline of both radiative and dynamical instability.

$dP_{\rm turb}/dP$, across the convection zone. Our approximation of slowly adapting turbulence appears, nevertheless, to be on safe ground since $v_{\rm turb}/v_{\rm sound}$ is so small everywhere.

5. LUMINOUS BLUE VARIABLES AND HYDROGEN-POOR WN STARS

As the effective temperature of a post–red-supergiant star increases, the outer envelope mass, δM, declines sharply. Using observed mass-loss rates for LBVs and for their close relatives, the hydrogen-poor WN stars (assumed to be post–red-supergiant stars), we have previously shown that f becomes a significant fraction of g when $T_e > 20{,}000$ K (Stothers 2002a). In these models, f was included explicitly. The consequent reduction of the effective gravity was found to lower the Eddington luminosity limit, leading to a greater tendency toward radiative instability as well as toward dynamical instability.

At the same time, the approach of the outer envelope to the Eddington limit means that the greatly reduced densities in the deep adiabatic layers must be compensated for by substantially increased mean turbulent velocities if convection is to carry enough flux to avoid radiative expulsion of the outer envelope (§ 1). Figure 3 displays the runs of several physical variables inside an extreme blue supergiant model of 21.6 $M_{\odot}$ with $\log T_e = 4.35$, calculated by assuming $\alpha_P = 1.4$ and $f/g = 0.16$. The run of density is found to be even flatter than in a yellow hypergiant model, the opacity bumps are more muted, and the ratio $v_{\rm turb}/v_{\rm sound}$ is very small in the helium convection zone owing to this zone's closeness to the stellar surface. On the other hand, very deep in the iron convection zone, $v_{\rm turb}/v_{\rm sound}$ shoots up to a supersonic value of 1.3. Since this zone is strongly convective (essentially adiabatic), convection is able to carry all of the flux in it without disruption of the outer envelope, as long as pulsations do not suppress the efficiency of convection. With a slight increase of f/g, however, convection finally becomes an inadequate carrier, even without any pulsations, and radiative instability formally breaks out in many of the layers of the outer envelope.

It is not possible at present to determine realistically the true physical conditions in the supersonic iron convection zone. First of all, the densities become so low that they fall outside the range of the available opacity tables. Furthermore, below some value of the density, it is physically necessary that $\delta \rightarrow 0$ as $\beta \rightarrow 0$, but this critical density is unknown. Then, standard mixing-length theory, which is very crude to begin with, fails even in concept when the mean turbulent velocity exceeds sound velocity. It is a reasonable assumption that supersonic turbulence would create shocks, which would then rapidly dissipate the large eddies and generate both acoustic energy and heat energy. Cox & Giuli (1968) assumed that this process would reduce $v_{\rm turb}$ to near equality with $v_{\rm sound}$, although Deng & Xiong (2001) argued that the smaller eddies produced by the shocks might possess subsonic velocities. In either case, convection would become less efficient, leading to a reduced convective flux and a reduced flux of turbulent kinetic energy. Although the turbulent pressure would also be decreased, sound waves would help to support the convection zone against gravity. On the other hand, because stellar envelope turbulence is of a forced nature rather than freely decaying, a supersonic state may well be maintained, as it apparently is in observed interstellar clouds (Larson 1979, 1981). A related problem is that the structure of strong turbulence in known physical situations is not random but consists, in part, of large-scale coherent features. Since these features probably arise from the self-regulating nature of turbulent viscosity (Stothers 2000; Canuto 2000), they are likely to be universal and therefore present also in luminous supergiant envelopes. For the moment, we must ignore the iron convection zone, and we proceed to focus on the largely radiative (or weakly convective) overlying layers.

5.1. *Radiative Instability*

The criterion for radiative instability, $L > L_E$, depends crucially on how the Eddington luminosity L_E is defined. Here we introduce the mass-loss acceleration and the turbulent pressure gradient in order to generalize Eddington's (1921) original derivation. Division of the equation of radiative transfer,

$$\frac{dP_{\rm rad}}{dr} = -\frac{\kappa\rho L}{4\pi c r^2} \ , \qquad (16)$$

into the full hydrostatic equilibrium equation (2) for $d(P_{\rm gas} + P_{\rm rad})/dr$, and making the approximation $M(r) = M$, and then finally integrating the resulting equation from the surface down to a layer at r yields

$$1 - \beta = \frac{\langle\kappa\rangle L}{4\pi c G M(1 - \psi - \xi)} \ , \qquad (17)$$

where

$$\psi = \frac{f}{g} \ , \quad \xi = -\frac{4\pi r^2}{g}\frac{dP_{\rm turb}}{dM(r)} \ , \qquad (18)$$

$$\frac{1}{\langle\kappa\rangle} = \frac{1}{(1 - \psi - \xi)P_{\rm rad}}\int_R^r \frac{1 - \psi - \xi}{\kappa}\frac{dP_{\rm rad}}{dr}\,dr \ . \qquad (19)$$

Since it is necessary for radiative stability that $\beta \geq 0$, the generalized Eddington luminosity must be

$$L_E = \frac{4\pi c G M(1 - \psi - \xi)}{\langle\kappa\rangle} \ . \qquad (20)$$

The *local* definition of L_E with $\langle\kappa\rangle$ replaced by the local value of κ applies only if the quantities κ, ψ, and ξ are all constant in the overlying layers or if the local density and temperature do not increase outward (Chandrasekhar 1939, p. 221). In our blue supergiant envelope models, the *nonlocal* definition of L_E is needed because even though κ is roughly constant in the envelope and ψ has been taken to be constant there, a density inversion occurs in the weakly convective helium convection zone. This inversion is due to a small opacity bump arising from second helium ionization. The effect on L_E of the turbulent pressure in this zone is slight, however, for the following reason. In the layers with maximum turbulent pressure, the ratio $v_{\rm turb}/v_{\rm sound}$ has a vanishing gradient. Therefore, the quantity δ is not only maximal but also locally constant. Consequently, the effective gravity ratio is a minimum,

$$1 - \psi - \xi = \frac{1 - fg^{-1}}{1 + \delta} \ . \qquad (21)$$

In the model of Figure 3, $\delta \leq 10^{-2}$, which is negligible in magnitude compared to unity. For our other extreme blue supergiant models, similar results for δ in the helium convection zone are found.

Throughout the tenuous envelopes of our most extreme models, L_E remains very close to L, a value that is fixed essentially by the underlying mass of the star. Above the iron convection zone, we find that radiative instability is controlled chiefly by the value of ψ. Under the assumption of constant ψ, it has previously been possible to derive for any stellar envelope model a critical value of ψ for which $L = L_E$ (Stothers 2002a). The originally derived values of ψ are now found to remain approximately valid even when turbulent pressure is included in the models.

Owing to the presence of a supersonic turbulent pressure in the iron convection zone, however, the models with L_E very close to L are not very securely established. If radiative instability in such models is governed largely by conditions inside the iron convection zone, instability could break out there while the overlying layers still have L_E well above L. In that case, the mass-loss rate inferred from equation (15) applied to the overlying layers might be too large. It then becomes possible, in theory, that supersonic turbulence in the iron convection zone could drive off the outer envelope without needing any help at all from the superincumbent layers.

5.2. *Dynamical Instability*

To test the model envelopes for dynamical instability, two different turbulent scenarios—slowly adapting turbulence in the helium convection zone and rapidly adapting turbulence in the iron convection zone—have been considered. After equations (2) and either (6)–(8) or (12)–(14) have been applied, only insignificant changes from our earlier results without turbulent pressure are obtained. There are two reasons for this.

First, the part of the envelope contributing most to the determination of dynamical stability or instability consists of the cooler layers situated above the iron convection zone, where the displacement amplitude $\delta r/r$ is large enough to be significant. This result can be understood by examining an approximate form of the instability criterion $\sigma^2 \leq 0$, which becomes exact if $\delta r/r$ = constant and $1 - \psi - \xi$ = constant (Stothers 1999):

$$\langle\Gamma_1\rangle = \frac{\int_{r^*}^R \Gamma_1 P\,d(r^3)}{\int_{r^*}^R P\,d(r^3)} \leq \frac{4}{3} \ . \qquad (22)$$

In this expression r^* refers to the base of the outer envelope. Assuming an envelope of uniform density, integration of the equation of hydrostatic equilibrium yields $P \approx GM\rho(1 - \psi - \xi)/r$ in layers well below the photosphere, whence $P\,d(r^3) \approx (3/2)GM\rho(1 - \psi - \xi)\,d(r^2)$. Consequently, the weighting factor for Γ_1 at great depth is proportional to the local surface area, which becomes very small when r^* is reached since $r^* \ll R$. In our envelope models, the weighting factor is found to be negligible for temperatures higher than 1×10^5 K (Fig. 3), and so the large turbulent pressure in the iron convection zone has little effect on the determination of dynamical stability of the envelope.

The second reason why turbulent pressure is ineffectual in our models is that in the convective layers where $\delta r/r$ is largest—specifically, in the helium convection zone that extends from the surface down to the layers with $T \approx 4 \times 10^4$ K—the turbulent pressure remains very small.

When the iron convection zone is not included in the calculations, the effect of the helium convection zone is one of a slight destabilization. On the other hand, the iron convection zone itself tends to slightly stabilize the envelope by raising the local value of Γ_1 modified to include turbulent pressure, viz.,

$$\Gamma_1 = \frac{32 + 40\delta + 5\delta^2}{24 + 27\delta + 3\delta^2} , \quad (23)$$

which applies whenever $\beta \ll 1$ (Stothers 2002b). These conclusions are quite general for our models of LBVs. Quantitatively, the net effect of the turbulent pressure turns out to be rather variable but is always so small that it can be easily canceled by a minor change of f/g (<20%).

It seems almost superfluous to worry about $P_{\rm turb}$ in a consideration of dynamical instability for these stars. The observational uncertainty of dM/dt is probably a factor of $\sim$2 (Vink & de Koter 2002), which translates into an uncertainty of a factor of 4 in f/g, according to equation (15). This circumstance is why we have, instead, approached the problem purely theoretically, by finding the value of f/g that leads to marginal dynamical instability and, from this, predicting the value of dM/dt (Stothers 2002a). Since the models are so sensitive to f/g owing to their closeness to the Eddington limit, other factors such as turbulent pressure become much less important in comparison. The rapid change of the predicted values of dM/dt with effective temperature ensures that a comparison with the observed values of dM/dt, despite observational errors, results in a well-determined estimate of the critical effective temperature for dynamical instability.

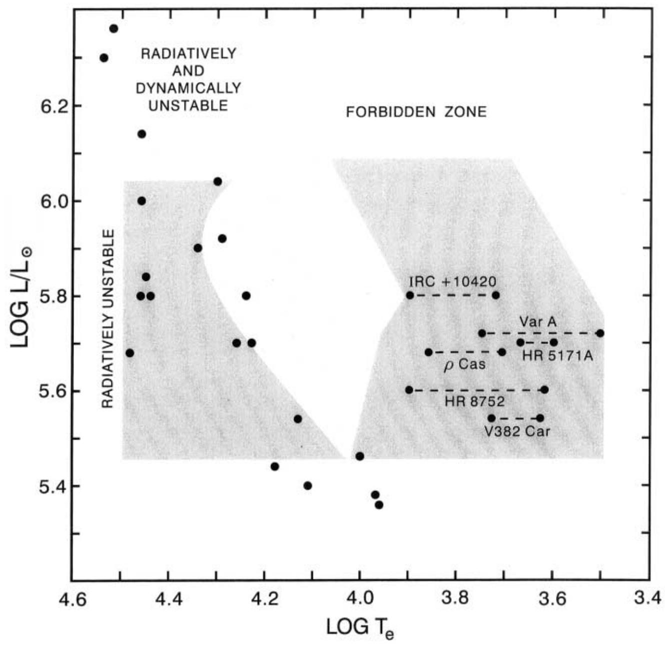

FIG. 4.—H-R diagram showing the theoretically predicted domains of dynamical instability for yellow hypergiants and for LBVs. Symbols denote individual observed stars: yellow hypergiants (*dots connected by dashed lines*) and LBVs at quiescence (*single dots*). The observations come from de Jager (1998) and van Genderen (2001).

6. CONCLUSION

Despite the fact that convective instability is widespread in the envelopes of luminous post–red-supergiant stars, the layers above the iron convection zone possess such low densities that radiative equilibrium prevails to a very close approximation. Furthermore, turbulent pressure in the hydrogen and helium convection zones turns out to be not as high as one might have expected on the basis of the initially anticipated large turbulent velocities there.

The iron convection zone, on the other hand, is deep enough to be nearly adiabatic and to be capable of generating large turbulent velocities. However, its great depth puts it below the layers where the question of dynamical stability or instability of the envelope is decided on the basis of the partial ionizations of hydrogen and helium. Moreover, its convective flows are probably able to transport all of the super-Eddington luminous flux, so that the question of radiative stability or instability of the envelope is most likely determined by the overlying, nearly radiative layers. This last conclusion, however, may not apply in the case of the bluest supergiants, because for them, turbulence near the bottom of the iron convection zone becomes supersonic, with very uncertain consequences. The cause of these remarkably high turbulent velocities is the need to transport a large convective flux in the presence of very low mass densities.

In general, by reducing the star's effective gravity, turbulent pressure tends to destabilize the envelope both radiatively and dynamically, especially if turbulence adapts only slowly to an imposed perturbation, but in all of our models except for the bluest ones, the effects of the turbulent pressure turn out to be very slight.

The major domains of dynamical instability on the H-R diagram that we published earlier for post–red-supergiant stars (Stothers & Chin 2001; Stothers 2002a) remain essentially valid as calculated. They are indicated here, schematically, in Figure 4 as the gray areas. The cool edge of the predicted LBV domain is determined by comparing the predicted rates of mass loss with the observed rates of mass loss, as discussed elsewhere (Stothers 2002a); turbulent pressure has a comparatively negligible effect on this cool edge. Turbulence, however, increases $\log T_e$ of the hot edge of the predicted yellow hypergiant domain by a small amount, but this shift is less than 0.1. Referring further to the figure, hot stars with $\log(L/L_\odot) > 6.0$ are found to be both dynamically and radiatively unstable, although our hydrostatic calculations are probably unreliable at such high luminosities; stars with $\log T_e > 4.5$ cannot be dynamically unstable, owing to an absence of hydrogen and helium ionization zones, but may be radiatively unstable; and stars with $\log(L/L_\odot) < 5.4$ also cannot be dynamically unstable, because their L/M ratios are too low. The most luminous stars of all are always blue and never reach the "forbidden zone" on the H-R diagram (Stothers & Chin 1999).

Superposed on Figure 4 are the observed locations of massive unstable post–main-sequence stars belonging to the Galaxy, M33, and the Large Magellanic Cloud, drawn primarily from the catalogs of de Jager (1998) and van Genderen (2001). The central void is real—a consequence of rapid evolution in this region (de Jager et al. 2001). Agreement with our theoretical predictions is seen to be close. Other points of agreement have been similarly pointed out in regard to the stellar masses, surface H and N abundances, mass-loss rates, instability cycles, and the luminosity function (Stothers & Chin 1996; Stothers 2002a) and need not be repeated here. We note only that the mass-loss rates in the models have been taken to be free parameters and have not been calculated hydrodynamically (as indeed they cannot yet be). We stress that no other free parameters of any significance exist in our theory.

Although turbulence may not have much of an effect on the outer envelopes of yellow hypergiants and of most LBVs (except for the bluest ones), this insignificance is emphatically not true of the dynamically active atmospheres of these stars (Nieuwenhuijzen & de Jager 1995; de Jager 1998; de Jager et al. 2001). The key, therefore, to unlocking the whole instability phenomenon will probably lie in a combined calculation of the turbulent atmosphere and the turbulent outer envelope in a realistic numerical simulation of the massive outflowing envelope.

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